

**BPUB 931**  
**NUMERICAL METHODS IN ECONOMICS**  
**Location: ????**  
**Meeting Time: Fall 2013, Wed. 3-6**  
**Professor Ulrich Doraszelski**

**CONTACT INFORMATION**

Office address: 1405 Steinberg Hall-Dietrich Hall  
Office hours: anytime by appointment  
Phone: (215) 898-7148  
Email: [doraszelski@wharton.upenn.edu](mailto:doraszelski@wharton.upenn.edu)

**COURSE DESCRIPTION**

The objective of this course is to introduce graduate students to computational approaches for solving economic models. We will formulate economic problems in computationally tractable form and use techniques from numerical analysis to solve them. Examples of computational techniques in the current economics literature as well as discuss areas where these techniques may be useful in future research will be discussed. We will pay particular attention to methods for solving dynamic optimization problems and computing equilibria of games. The substantive applications will cover a wide range of problems including industrial organization, game theory, macroeconomics, finance, and econometrics.

**READINGS**

Textbook:

- Judd, K. (1998) "Numerical Methods in Economics" MIT Press

Books on Reserve (Lippincott Library Reserve in Van Pelt):

- Chiang, A. (1992) "Elements of Dynamic Optimization" McGraw-Hill
- Zangwill, W. & Garcia, C. (1981) "Pathways to Solutions, Fixed Points, and Equilibria" Prentice-Hall
- Miranda, M. & Fackler, P. (2002) "Applied Computational Economics and Finance" MIT Press
- Stokey, N. & Lucas, R. (1989) "Recursive Methods in Economics Dynamics" Harvard University Press

Papers:

The course reader is available through study.net. Most papers can also be downloaded freely from the Penn Libraries.

Course Webpage:

Additional materials such as lecture notes and problem sets will be posted on Canvas.

## **PROBLEM SETS AND TERM PAPER**

Your course grade will be based on the problem sets (50%) and on the term paper (50%).

### Problem Sets and In-Class Exercises:

Doing computation is the only way to learn computation. The problem sets are designed to get you to apply various numerical methods. You are welcome to work in groups of two or three. More generally, I strongly encourage you to discuss the problem sets with your classmates. Classmates are also a valuable source for tips on programming and general computer advice.

A write-up of your solution (one write-up per group) is due in class on the date indicated on the problem set. The write-up should

- provide a brief and clear verbal description of the methods employed;
- use tables and/or graphs to describe your results;
- contain your computer programs in an appendix.

Just handing in computer programs and their output is not acceptable. Please remember to indicate the names of all group members on the write-up.

Some of the exercises will be discussed in class. These exercises are clearly labeled on the problem set and in the class schedule. Prior to each session work through the assigned readings and do the in-class exercises. Depending on the session I may assign different in-class exercises to different students. During class I will call on one or more students to present their solution to the in-class exercises. You will explain your solution to your classmates and illustrate it on the classroom computer. This means that you will have to bring your computer programs to class on a memory stick or your laptop.

Note that the goal is not to solve each and every exercise; instead the problem sets are meant to give you an opportunity to practice solving economic models on a computer.

### Term Paper:

You have considerable freedom to follow your interests in choosing a topic for a term paper. The paper should

- motivate and formulate a research question;
- explain why numerical methods are needed to answer the question;
- present a model;
- discuss a computational strategy for solving the model;
- present the results;
- evaluate the accuracy of the results.

The term paper should be at most 15 double-spaced pages excluding tables, graphs, and appendices (12 point font, 1 inch margins).

I urge you to start thinking of a topic early in the semester. The term paper could involve replicating a computational paper. Another way to develop a topic is to think of a

theoretical paper that does not compute any (except perhaps trivial) examples and to compute a small set of interesting examples. Ideally, however, the term paper would ask (and answer) an original research question. Once you have identified a topic, please write a short (at most one page) proposal. Email the proposal to me and make an appointment to discuss the suitability of the topic. I expect all students to have discussed a topic with me before Fri., October 25, 2013.

The term paper itself should formally define the problem and emphasize the numerical methods used to solve it (pros and cons of the chosen approach, available alternative approaches). It should further include an error analysis (compute and verify).

You will give a short presentation of your term paper in the last session. The goal is to help you organize your thoughts and obtain feedback from your classmates.

The final version of the paper is due Fri., December 20, 2013.

## **SOFTWARE AND HARDWARE**

There is no point in re-inventing the wheel, and I encourage you to use canned code whenever possible. The web is a valuable resource of ready-made code which is often written by professional programmers and/or applied mathematicians and has been debugged and optimized over many years. However, part of this course is to acquaint you with different numerical methods and programming a method is the best way to understanding how it works. Using canned code is not. I ask you to rely on your own judgment in making this decision. As a rule of thumb I suggest that in a session on optimization, say, you do not use an existing optimization routine but instead code your own.

### Matlab:

The default computer language for this course is Matlab. The complete documentation of Matlab and its toolboxes can be freely downloaded at [www.mathworks.com](http://www.mathworks.com).

A tutorial to get you started and programming tips are available at

[www.mathworks.com/access/helpdesk/help/pdf\\_doc/matlab/getstart.pdf](http://www.mathworks.com/access/helpdesk/help/pdf_doc/matlab/getstart.pdf)

[www.mathworks.com/access/helpdesk/help/pdf\\_doc/matlab/programming\\_tips.pdf](http://www.mathworks.com/access/helpdesk/help/pdf_doc/matlab/programming_tips.pdf)

Wharton has a site license for Matlab that covers Wharton-owned computers. For access please contact Kenny Davis.

Mario Miranda and Paul Fackler have written a Matlab toolbox called CompEcon. The toolbox includes equation solvers and optimization routines, a set of routines for function approximation, a set of numerical integration routines, routines for solving ordinary differential equations (both initial and boundary value problems), and routines for solving discrete- and continuous-time dynamic programming problems. The toolbox can be freely downloaded at [www4.ncsu.edu/~pfackler/compecon/toolbox.html](http://www4.ncsu.edu/~pfackler/compecon/toolbox.html). Its use is explained in detail in Miranda & Fackler's textbook (on reserve).

### Gambit:

Gamit is a software package for game theory and can be freely downloaded at <http://www.gambit-project.org/doc/index.html>.

#### Linux Grid:

Wharton has a linux grid that you can use for the purposes of this course and your own research. I encourage you to read up on it at <http://unix.wharton.upenn.edu/> and to request an account. Amongst other software packages the grid has Matlab. The CompEcon toolbox is installable by any user by running `/usr/local/demo/matlab/setup_compecon.sh`.

#### **SCHEDULING**

There are no classes on August 28, 2013 and October 30, 2013. There is a make-up class on Fri., October 25, 2013 (location TBA).

## CLASS SCHEDULE

(subject to change)

### SESSION 1: SEPTEMBER 4, 2013

#### Part 1: Introduction

##### Introduction to Numerical Methods

###### Topics:

- Course mechanics
- Roles of computation in economic analysis:
  - Generate hypothesis and counterexamples
  - Assess quantitative significance
  - Relax “technical” assumptions
  - Analyze complex models
- Deductive vs. computational theory
- A toolkit

###### Readings:

- Judd, Chap. 1
- Judd, K. (1997) “Computational Economics and Economic Theory: Substitutes or Complements?” *Journal of Economic Dynamics and Control* 21 (6) pp. 907-942
- Kubler, F. (2007) “Approximate Generalizations and Computational Experiments” *Econometrica* 75 (4) pp. 967-992 (optional)

###### Notes:

- Come to class prepared to name at least one paper that has (or should have) used numerical methods. What is (or would be) the value added by numerical methods?

##### Elementary Concepts of Numerical Analysis

###### Topics:

- Computer arithmetic
- Error analysis:
  - Sources of error
  - Error propagation
  - Controlling the error
  - Compute and verify
- Algorithmic complexity: speed and memory

###### Readings:

- Judd, Chap. 2

### SESSION 2: SEPTEMBER 11, 2013

## **Part 2: Basic Concepts**

### **Linear Equations**

#### Topics:

- Direct methods: Backsubstitution, LU and other decompositions
- Error analysis and condition numbers
- Iterative methods: Gauss-Jacobi and Gauss-Seidel
- Acceleration and stabilization methods

#### Readings:

- Judd, Chap. 3

### **Optimization**

#### Topics:

- Comparison methods
- Newton's method
- Direction set methods
- Constrained optimization

#### Readings:

- Judd, Chap. 4, Sections 4.1-4.7

#### Notes:

- Problem Set 1 due

## **SESSION 3: SEPTEMBER 18, 2013**

### **Optimization**

#### Topics:

- Applications:
  - Portfolio choice (Exercise 3-1)
  - Life-cycle consumption (Exercise 3-2)
  - Principal-agent and adverse selection problems (Exercise 3-3)
  - Normal form games with discrete actions (Exercise 3-4)
- Practical considerations:
  - Nature of problem and choice of algorithm
  - Local and global optima
  - Hot starts and cold starts
  - Canned code

#### Readings:

- Judd, Chap. 4, Sections 4.8-4.14

- McCullough, B. & Vinod, H. (2003) “Verifying the Solution from a Nonlinear Solver: A Case Study” *The American Economic Review* 93 (3) pp. 873-892 (optional)
- Shachar, R. & Nalebuff, B. (2004) “Verifying the Solution from a Nonlinear Solver: A Case Study: Comment” *The American Economic Review* 94 (1) pp. 382-390 (optional)
- McCullough, B. & Vinod, H. (2004) “Verifying the Solution from a Nonlinear Solver: A Case Study: Reply” *The American Economic Review* 94 (1) pp. 391-396 (optional)
- Drukker, D. & Wiggins, V. (2004) “Verifying the Solution from a Nonlinear Solver: A Case Study: Comment” *The American Economic Review* 94 (1) pp. 397-399 (optional)

Notes:

- In-class discussion of Exercises 3-1, 3-2, 3-3, and 3-4 from Problem Set 3

### **Nonlinear Equations**

Topics:

- Bisection method
- Gauss-Jacobi and Gauss-Seidel methods
- Fixed-point iteration
- Newton’s method
- Powell’s hybrid method

Readings:

- Judd, Chap. 5, Sections 5.1-5.7

Notes:

- Problem Set 2 due

### **SESSION 4: SEPTEMBER 25, 2013**

### **Nonlinear Equations**

Topics:

- Applications:
  - Competitive general equilibrium (Exercise 4-1)
  - Asset pricing (Exercise 4-2)
  - Normal form games with continuous actions (Exercise 4-3)
- Continuation and homotopy methods

Readings:

- Judd, Chap. 5, Sections 5.8-5.12
- Zangwill, W. & Garcia, C. (1981), Chap. 1, 2 (less technical, optional)

- Allgower, E. & Georg, K. (1993) “Continuation and Path Following” *Acta Numerica* 2 pp. 1-64 (more technical, optional)

Notes:

- In-class discussion of Exercises 4-1, 4-2, and 4-3 from Problem Set 4

**Finite Games**

Topics:

- Overview of methods for normal and extensive form games
- Gambit tutorial
- Application to product choice and estimation of finite games

Readings:

- McKelvey, R. & McLennan, A. (1996) “Computation of Equilibria in Finite Games” in Amman, H., Kendrick, D. & Rust, J. (eds.): *Handbook of Computational Economics, Volume 1* North-Holland pp. 88-142 (very technical, don’t get bogged down by the details)
- Doraszelski, U. & Draganska, M. (2006) “Market Segmentation Strategies of Multiproduct Firms” *Journal of Industrial Economics* 54 (1) pp. 125-149 (skim to see how Gambit is used)
- Bajari, P., Hong, H. & Ryan, S. (2010) “Identification and Estimation of Discrete Games of Complete Information” *Econometrica* 78 (5) pp. 1529-1568 (skim to see how Gambit is used)
- von Stengel, B. (2002) “Computing Equilibria for Two-Person Games” in Aumann, R. & Hart, S. (eds.): *Handbook of Game Theory with Economic Applications, Volume 3* North-Holland pp. 1725-1759 (optional)

Notes:

- Problem Set 3 due

**SESSION 5: OCTOBER 2, 2013**

**Introduction to the Wharton Grid**

Guest speaker: Hugh MacMullan, Wharton IT

Notes:

- Problem Set 4 due

**SESSION 6: OCTOBER 9, 2013**

**Function Approximation**

Topics:

- Approximation vs. statistics



- Local approximation methods:
  - Taylor series expansion
  - Padé expansion
- Global approximation methods:
  - $L^p$  approximation: orthogonal polynomials
  - Interpolation methods: Lagrange, Hermite, Chebyshev, and piecewise polynomial (spline) interpolation
  - Regression methods: Chebyshev regression

#### Readings:

- Judd, Chap. 6, Sections 6.1-6.10

### **Function (and Set) Approximation**

#### Topics:

- Shape-preserving methods
- Multidimensional methods:
  - Tensor products and complete polynomials
  - Chebyshev regression
  - Finite element methods
  - Neural networks
- Set approximation and application to repeated games

#### Readings:

- Judd, Chap. 6, Sections 6.11-6.15
- Judd, K., Yeltekin, S. & Conklin, J. (2003) “Computing Supergame Equilibria” *Econometrica* 71 (4) pp. 1239-1254
- Abreu, D., Pearce, D. & Stacchetti, E. (1990) “Toward a Theory of Discounted Repeated Games with Imperfect Monitoring” *Econometrica* 58 (5) pp. 1041-1063 (optional)

### **SESSION 7: OCTOBER 16, 2013**

### **Numerical Integration and Differentiation**

#### Topics:

- Newton-Cotes formulas
- Gaussian formulas
- Multidimensional integration: product rules and nonproduct rules
- Numerical differentiation

#### Readings:

- Judd, Chap. 7

### **Monte Carlo and Quasi-Monte Carlo Methods**

Topics:

- Pseudorandom number generation
- Monte Carlo integration
- Stochastic search and stochastic approximation
- Quasi-Monte Carlo integration
- Application: Portfolio choice (Exercises 6-2, 6-3, 6-4, and 6-5)

Readings:

- Judd, Chap. 8, 9

Notes:

- In-class discussion of Exercises 6-2, 6-3, 6-4, and 6-5 from Problem Set 6
- Problem Set 5 due

**SESSION 8: OCTOBER 23, 2013**

**Part 3: Methods for Functional Equations**

**Finite-Difference Methods**

Topics:

- Initial value problems:
  - Euler methods
  - Runge-Kutta methods
- Boundary value problems: shooting
- Application to finite- and infinite-horizon optimal control problems
- Integral equations

Readings:

- Judd, Chap. 10
- Chiang, A. (1992), Chap. 7, Sections 7.1-7.4, Chap. 8, Section 8.2, Chap. 9, Section 9.1 (background material on optimal control problems)

**Projection Methods**

Topics:

- Choice of basis functions
- Choice of approximation degree
- Choice of projection conditions
- Computation of solution
- Verification of solution

Readings:

- Judd, Chap. 11

Notes:

- Problem Set 6 due

**SESSION 9: OCTOBER 25, 2013**

**Projection Methods**

Topics:

- Applications:
  - Signaling (Exercise 7-3)
  - Learning-by-doing (Exercise 7-4)
  - Life-cycle consumption (Exercise 7-5)
  - Optimal growth (Exercise 7-6)

Readings:

- Spence, M. (1974) “Competitive and Optimal Response to Signals: An Analysis of Efficiency and Distribution” *Journal of Economic Theory* 7 (3) pp. 296-332. (optional)

**Numerical Dynamic Programming**

Topics:

- Discrete-time dynamic programming
- Continuous-time dynamic programming
- Methods for finite-state problems:
  - Value function iteration
  - Policy function iteration
  - Gaussian acceleration methods
- Methods for continuous-state problems:
  - Discretization
  - Parametric approximation methods
  - Projection methods

Readings:

- Judd, Chap. 12
- Lecture notes “Review of Dynamic Programming”
- Stokey, N. & Lucas, R. (1989), Chap. 3, 4, 9 (background material on dynamic programming)

Notes:

- In-class discussion of Exercises 7-3, 7-4, 7-5, and 7-6 from Problem Set 7

**SESSION 10: NOVEMBER 6, 2013**

**Perfect Foresight Models**

Topics:

- Time domain methods:
  - Tatonnement method
  - Fixed-point iteration
  - Newton's method
  - Parametric path method
  - Fair-Taylor method
- Recursive methods:
  - Projection methods
  - Time iteration
  - Fixed-point iteration
- Applications:
  - OLG (Exercise 8-2)
  - Optimal growth (Exercise 8-3)
  - Time-consistent taxation (Exercise 8-4)

Readings:

- Judd, Chap. 16

**TBA**

Guest speaker: TBA

Notes:

- In-class discussion of Exercises 8-2, 8-3, and 8-4 from Problem Set 8
- Problem Set 7 due

**SESSION 11: NOVEMBER 13, 2013**

**Rational Expectations Models**

Topics:

- Recursive methods:
  - Simulation method
  - Integral equation method
  - Projection methods
  - Time iteration
  - Fixed-point iteration
  - Wright-Williams smoothing
- Applications:
  - Asset pricing (Exercise 9-1)
  - Commodity storage (Exercises 9-2 and 9-3)

Readings:

- Judd, Chap. 17

Notes:

- In-class discussion of Exercises 9-1, 9-2, and 9-3 from Problem Set 9

**Heterogeneous Agent Models**

Topics:

- Optimal growth
- Incomplete asset markets

Readings:

- Gaspar, J. & Judd, K. (1997) "Solving Large-Scale Rational-Expectations Models" *Macroeconomic Dynamics* 1 (1) pp. 45-75 (Sections 1, 2, 4-7)
- Judd, K., Kubler, F. & Schmedders, K. (2000) "Computing Equilibria in Infinite-Horizon Finance Economies: The Case of One Asset" *Journal of Economic Dynamics and Control* 24 (5-7) pp. 1047-1078
- Judd, K., Kubler, F. & Schmedders, K. (2003) "Computational Methods for Dynamic Equilibria with Heterogeneous Agents" in Dewatripont, M., Hansen, L. & Turnovsky, S. (eds.): *Advances in Economics and Econometrics: Theory and Applications, Eighth World Congress, Volume 3* Cambridge University Press pp. 243-290 (Sections 6-9)
- Kubler, F. & Schmedders, K. (2003) "Stationary Equilibria in Asset-Pricing Models with Incomplete Markets and Collateral" *Econometrica* 71 (6) pp. 1767-1793 (optional)
- Kubler, F. & Schmedders, K. (2005) "Approximate Versus Exact Equilibria in Dynamic Economies" *Econometrica* 73 (4) pp. 1205-1236 (optional)

Notes:

- Problem Set 8 due

**SESSION 12: NOVEMBER 20, 2013**

**Introduction to Dynamic Games with Finite States**

Topics:

- From dynamic programming to dynamic games
- Value function iteration approach
- Markov-perfect industry dynamics
- Existence, purification, and multiplicity of equilibrium
- Open questions

Readings:

- Doraszelski, U. & Satterthwaite, M. (2010) "Computable Markov-Perfect Industry Dynamics" *Rand Journal of Economics* 41 (2) pp. 215-243

- Pakes, A. & McGuire, P. (1994) “Computing Markov-Perfect Nash Equilibria: Numerical Implications of a Dynamic Differentiated Product Model” *Rand Journal of Economics* 25 (4) pp. 555-589
- Ericson, R. & Pakes, A. (1995) “Markov-Perfect Industry Dynamics: A Framework for Empirical Work” *Review of Economic Studies* 62 pp. 53-92 (optional)
- Pakes, A. & McGuire, P. (2001) “Stochastic Algorithms, Symmetric Markov Perfect Equilibrium, and the 'Curse' of Dimensionality” *Econometrica* 69 (5) pp. 1261-1281 (optional)

## **Topics in Dynamic Games with Finite States I**

### Topics:

- Alternative formulations:
  - Continuous-time stochastic games with finite states
  - Discrete-time stochastic games with alternating moves and finite states

### Readings:

- Doraszelski, U. & Judd, K. (2011) “Avoiding the Curse of Dimensionality in Dynamic Stochastic Games” *Quantitative Economics* 3(1) pp. 53-93
- Doraszelski, U. & Judd, K. (2007) “Dynamic Stochastic Games with Sequential State-to-State Transitions”

### Notes:

- Problem Set 9 due

## **SESSION 13: NOVEMBER 27, 2013**

## **Topics in Dynamic Games with Finite States II**

### Topics:

- Applications:
  - Quality ladder model (Exercise 10-3)
  - Learning-by-doing model (Exercise 10-4)
- Computing all equilibria: Homotopy method

### Readings:

- Besanko, D., Doraszelski, U., Kryukov, Y. & Satterthwaite, M. (2010) “Learning-by-Doing, Organizational Forgetting, and Industry Dynamics” *Econometrica* 78 (2) pp. 453-508
- Borkovsky, R., Doraszelski, U., & Kryukov, Y. (2010) “A User’s Guide to Solving Dynamic Stochastic Games Using the Homotopy Method” *Operations Research* 58 (4) pp. 1116-1132
- Cabral, L. & Riordan, M. (1994) “The Learning Curve, Market Dominance, and Predatory Pricing” *Econometrica* 62 (5) pp. 1115-1140 (optional)

Notes:

- In-class discussion of Exercises 10-3 and 10-4 from Problem Set 10

**Dynamic Games with Continuous States**

Topics:

- Discrete-time dynamic games with continuous states
- Continuous-time dynamic games with continuous states
- Closed-form solutions:
  - Linear-quadratic games
  - Linear-state games
  - Exponential games
- Projection methods
- Application: R&D races

Readings:

- Judd, Chap. 12, Section 12.6
- Doraszelski, U. (2003) "An R&D Race with Knowledge Accumulation" *Rand Journal of Economics* 34 (1) pp. 20-42
- Reinganum, J. (1981) "Dynamic Games of Innovation" *Journal of Economic Theory* 25 pp. 21-41 (optional)
- Vedenov, D. & Miranda, M. (2001) "Numerical Solution of Dynamic Oligopoly Games with Capital Investment" *Economic Theory* 18 pp. 237-261 (optional)
- Reynolds, S. (1987) "Capacity Investment, Preemption and Commitment in an Infinite Horizon Model" *International Economic Review* 28 (1) pp. 69-88 (optional)

**SESSION 14: DECEMBER 4, 2013**

**Wrap-up and Student Presentations**

Topics:

- Wrap-up
- Student presentations

Notes:

- Problem Set 10 due